International Journal of Theoretical Physics, Vol. 36, No. 1, 1997

Darboux Transformations and Supersymmetrization Procedures

N. Debergh[1,2] and C. Gotti[1]

Received April 2, 1996

We discuss the Darboux transformations—or, in an equivalent way, the factorization method—in connection with two procedures of supersymmetrization available in two- and three-dimensional spaces, namely the standard and the spin–orbit coupling procedures.

1. INTRODUCTION

Since Darboux's contribution (Darboux, 1882), the construction of solutions of the Sturm–Liouville equation

$$-\psi_{xx} + u(x)\psi = \lambda\psi \tag{1.1}$$

where ψ_x stands for the derivative of ψ with respect to x, has been well known. Indeed, if ψ_1 is a particular solution of (1.1) (corresponding to the eigenvalue λ_1), it is easy to convince oneself that the function

$$\psi_{[1]} = \frac{W(\psi_1, \psi)}{\psi_1}, \qquad W(\psi_1, \psi) \equiv \psi_1\psi_x - \psi_{1x}\psi \tag{1.2}$$

given in terms of the usual Wronskian determinant, is a solution of (1.1), where the function $u(x)$ has been replaced by

$$u_{[1]}(x) = u(x) - 2(\ln \psi_1)_{xx} \tag{1.3}$$

In other words, Darboux's theorem declares that the Sturm–Liouville equation is invariant under the transformations $\psi \to \psi_{[1]}$ and $u \to u_{[1]}$.

[1] Theoretical and Mathematical Physics, Institute of Physics, B.5, University of Liège, B-4000-Liege 1, Belgium.
[2] Chercheur, Institut Interuniversitaire des Sciences Nucléaires, Brussels, Belgium.

0020-7748/97/0100-0111$12.50/0 © 1997 Plenum Publishing Corporation

Equation (1.1) plays a prominent role in quantum mechanics: it is nothing else but the time-independent Schrödinger equation, while ψ, $\frac{1}{2}u$, and $\frac{1}{2}\lambda$ are the wavefunction, the potential, and the energy, respectively. In that context, the Darboux transformations (1.2) and (1.3) have been exploited in the equivalent form

$$B^{-}H = H_{[1]}B^{-} \tag{1.4}$$

called the factorization method (Schrödinger, 1940). In (1.4) we have introduced the operators

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}u \tag{1.5a}$$

$$H_{[1]} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}u_{[1]} \tag{1.5b}$$

and

$$B^{-} = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} - (\ln \psi_1)_x\right) = \frac{1}{\sqrt{2}}(ip - (\ln \psi_1)_x) \tag{1.5c}$$

Such developments have been reconsidered in the recent literature (Matveev and Salle, 1991) because of their connection with supersymmetric quantum mechanics (Witten, 1981). In fact, one can see that the Witten superalgebra characterized by the anticommutation relations

$$\{Q, Q\} = 0, \qquad \{Q, Q^{\dagger}\} = H_{SS} \tag{1.6a}$$

and the commutation relation [equivalent to (1.4)]

$$[H_{SS}, Q] = 0 \tag{1.6b}$$

where H_{SS} is the (self-adjoint) supersymmetric Hamiltonian, can be realized with

$$H_{SS} = \begin{pmatrix} H & 0 \\ 0 & H_{[1]} \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ B^{-} & 0 \end{pmatrix} \tag{1.7}$$

if

$$u = [(\ln \psi_1)_x]^2 + (\ln \psi_1)_{xx} \tag{1.8a}$$

$$u_{[1]} = [(\ln \psi_1)_x]^2 - (\ln \psi_1)_{xx} \tag{1.8b}$$

In these supersymmetric developments, the function $(\ln \psi_1)$ is usually referred to as the superpotential (Witten, 1981).

Let us finally mention that the generalization of (1.4) to the (physically interesting) contexts of two and three space dimensions has already been achieved (Andrianov *et al.*, 1984, 1985), but without any connection to a specific procedure of supersymmetrization. The purpose of this paper is thus to relate the results of Andrianov *et al.* (1984, 1985) to such a procedure, i.e., the standard procedure (Witten, 1981), and to extend these results to another possible procedure, i.e., the spin–orbit coupling procedure (Balantekin, 1985; Beckers *et al.*, 1987). As we will show, in the three-dimensional case, the latter is much more interesting, physically speaking, because it includes (in contrast to the standard one) the concept of "*antiparticles*" (Moshinsky *et al.*, 1990).

In Section 2 we discuss the Andrianov *et al.* results in the context of *two* space dimensions, relate them to the standard procedure, and extend them to the case of the spin–orbit coupling procedure. A similar analysis is introduced in Section 3 for the context of *three* space dimensions and the differences between the two procedures are discussed.

2. THE CONTEXT OF TWO SPACE DIMENSIONS

If, by analogy with (1.5c), we introduce the operators

$$B_l^- = \frac{1}{\sqrt{2}}[ip_l - (\ln \psi_1)_{xl}], \qquad l = 1, 2 \tag{2.1}$$

we can, after Andrianov *et al.* (1984, 1985), generalize the condition (1.4) as

$$B_l^- H = H_{[1]lm} B_m^- \tag{2.2a}$$

and

$$H_{[2]} C_l^- = C_m^- H_{[1]ml}, \qquad l, m = 1, 2 \tag{2.2b}$$

if

$$C_l^- = \epsilon_{lm} B_m^- \tag{2.3}$$

where ϵ stands for the (antisymmetric) Levi-Cevita pseudotensor and the sum on repeated indices is understood.

In a parallel way to the one-dimensional case, we can rewrite the relations (2.2) in a matricial form (1.6b) with the identifications

$$H_{SS} = He_{1,1} + H_{[1]lm} e_{l+1,m+1} + H_{[2]} e_{4,4} \tag{2.4}$$

$$Q = B_l^- e_{l+1,1} + C_l^- e_{4,l+1} \tag{2.5}$$

The notation $e_{k,l}$ refers to (4 by 4 in this case) matrices containing zeros everywhere except unity at the intersection of the kth row and the lth column. In order to satisfy all the Witten requirements, we also have to require the anticommutators (1.6a), i.e.,

$$[B_1^-, B_2^-] = 0 \tag{2.6}$$

and

$$H = B_l^+ B_l^- \tag{2.7a}$$

$$H_{[1]lm} = H\delta_{lm} + [B_l^-, B_m^+] \tag{2.7b}$$

$$H_{[2]} = B_l^- B_l^+ \tag{2.7c}$$

The condition (2.6), which in terms of the superpotential is

$$(\ln \psi_1)_{x_1 x_2} = 0 \tag{2.8}$$

is automatically obeyed for central problems including the harmonic oscillator interaction, which corresponds to

$$\ln \psi_1 = \frac{1}{2} \omega x_l^2 \tag{2.9}$$

where ω is the angular frequency of that oscillator. In this case, the supersymmetric Hamiltonian (2.4) is simply

$$H_{SS} = \frac{1}{2} p_l^2 + \frac{1}{2} \omega^2 x_l^2 + \omega(e_{1,1} - e_{4,4}) \tag{2.10}$$

and we can thus conclude that the Andrianov *et al.* constraints (2.2) are typical of the standard (Witten, 1981) supersymmetrization procedure.

As is well known (Balantekin, 1985; Beckers *et al.*, 1987), there exists another procedure leading to another possible supersymmetric oscillator associated with the 2 by 2 matrix

$$H_{SS}^{SO} = \frac{1}{2} p_l^2 + \frac{1}{2} \omega^2 x_l^2 + \omega(e_{1,1} - e_{2,2}) + \omega L_3 \tag{2.11}$$

The introduction of the third component of the angular momentum is a specific feature of this procedure, which is called the spin–orbit coupling procedure. The corresponding Q is given by

$$Q = (B_1^- + iB_2^-)e_{2,1} \tag{2.12}$$

and it is straightforward to rewrite the commutator (1.6b) in the form

$$(B_1^- + iB_2^-)H^{SO} = H_{[1]}^{SO}(B_1^- + iB_2^-) \tag{2.13}$$

analogous to (1.4) if

$$H_{SS}^{SO} = H^{SO}e_{1,1} + H_{[1]}^{SO}e_{2,2}$$
$$= (B_1^+ - iB_2^+)(B_1^- + iB_2^-)e_{1,1} + (B_1^- + iB_2^-)(B_1^+ - iB_2^+)e_{2,2} \quad (2.14)$$

We can thus conclude that the Darboux transformations also hold for this spin–orbit coupling procedure.

Another way to confirm this conclusion is to try to insert the spin–orbit coupling features inside the standard ones. For this aim, we propose the following identifications:

$$H = H^{SO}, \qquad H_{[2]} = H_{[1]}^{SO} \quad (2.15)$$

$$D_1^- = D_2^- = \frac{1}{\sqrt{2}}(B_1^- + iB_2^-) \quad (2.16)$$

Then, if we consider the operators (2.5) where we have replaced [also in (2.2) and (2.3)] the B_l^- by the D_l^- defined in (2.16), we satisfy the conditions (2.2) with

$$H_{[1]11} = H_{[1]22} = \{D_1^-, D_1^+\} \quad (2.17a)$$

and

$$H_{[1]12} = H_{[1]21} = [D_1^-, D_1^+] \quad (2.17b)$$

Consequently, we confirm the previous conclusion. Moreover, when the oscillator interaction (2.9) is under study, the operators (2.17) lead to [see (2.4)]

$$H_{SS} = \frac{1}{2}p_l^2 + \frac{1}{2}\omega^2 x_l^2 + \omega(e_{1,1} - e_{2,3} - e_{3,2} - e_{4,4}) + \omega L_3 \quad (2.18)$$

When the unitary transformation

$$U = e_{1,1} + e_{4,4} + \frac{1}{\sqrt{2}}(e_{2,2} - e_{2,3} - e_{3,2} - e_{3,3}) \quad (2.19)$$

is applied to this Hamiltonian (2.18), it becomes

$$H_{SS} = H_{SS}^{SO} \otimes I_2 \quad (2.20)$$

where I_2 is the identity operator in the two-dimensional space, and the inclusion of the spin–orbit coupling case in the standard one is particularly clear.

3. THE CONTEXT OF THREE SPACE DIMENSIONS

The operators B_l^- we consider now are still defined by (2.1) up to the fact that the index l runs from 1 to 3. The relations (2.2) when extended to a three-dimensional space are

$$B_l^- H = H_{[1]lm} B_m^- \tag{3.1a}$$

$$H_{[2]lk} C_{km}^- = C_{lk}^- H_{[1]km} \tag{3.1b}$$

$$H_{[3]} B_l^- = B_m^- H_{[2]ml} \tag{3.1c}$$

$$C_{km}^- = \epsilon_{kml} B_l^-, \qquad k, l, m = 1, 2, 3 \tag{3.2}$$

If we put the conditions (3.1) in a matricial form (1.6b), we are led to

$$H_{SS} = H e_{1,1} + H_{[1]lm} e_{l+1,m+1} + H_{[2]lm} e_{l+4,m+4} + H_{[3]} e_{8,8} \tag{3.3}$$

$$Q = B_l^- (e_{l+1,1} - e_{8,l+4}) + C_{lm}^- e_{m+4,l+1} \tag{3.4}$$

Once again, in order to complete the supersymmetric requirements, we have to add

$$[B_j^-, B_l^-] = 0 \tag{3.5}$$

and

$$H = B_l^+ B_l^- \tag{3.6a}$$

$$H_{[1]lm} = B_l^- B_m^+ + C_{lk}^+ C_{mk}^- \tag{3.6b}$$

$$H_{[2]lm} = B_l^+ B_m^- + C_{lk}^- C_{mk}^+ \tag{3.6c}$$

$$H_{[3]} = B_l^- B_l^+ \tag{3.6d}$$

The dimension of the (8 by 8) matrices and the particular context of the oscillator interaction (2.9) imply that

$$H_{SS} = \frac{1}{2} p_l^2 + \frac{1}{2} \omega^2 x_l^2 + \frac{\omega}{2} (3e_{1,1} + e_{2,2} + e_{3,3} + e_{4,4} - e_{5,5} - e_{6,6} - e_{7,7} - 3e_{8,8}) \tag{3.7}$$

and this leads us to the conclusion that these conditions (3.1) are actually relevant to the standard procedure (Witten, 1981).

The spin–orbit coupling version of (3.7) is (Balantekin, 1985; Beckers *et al.*, 1987)

$$H_{SS}^{SO} = \frac{1}{2} p_l^2 + \frac{1}{2} \omega^2 x_l^2 + \frac{3\omega}{2} (e_{1,1} + e_{2,2} - e_{3,3} - e_{4,4}) + \omega(\mathbf{L} \cdot \boldsymbol{\sigma}) \otimes (e_{1,1} - e_{2,2}) \tag{3.8}$$

and we deal with 4 by 4 matrices only (the σ's are the usual Pauli matrices). The associated generator Q is

$$Q = B_l^- \sigma_l \otimes e_{2,1} \tag{3.9}$$

and implies the relations

$$B_3^- H_{11}^{SO} + (B_1^- - iB_2^-)H_{21}^{SO} = H_{[1]11}^{SO} B_3^- + H_{[1]12}^{SO}(B_1^- + iB_2^-) \tag{3.10a}$$

$$B_3^- H_{12}^{SO} + (B_1^- - iB_2^-)H_{22}^{SO} = H_{[1]11}^{SO}(B_1^- - iB_2^-) - H_{[1]12}^{SO} B_3^- \tag{3.10b}$$

$$(B_1^- + iB_2^-)H_{11}^{SO} - B_3^- H_{21}^{SO} = H_{[1]21}^{SO} B_3^- + H_{[1]22}^{SO}(B_1^- + iB_2^-) \tag{3.10c}$$

$$(B_1^- + iB_2^-)H_{12}^{SO} - B_3^- H_{22}^{SO} = H_{[1]21}^{SO}(B_1^- - iB_2^-) - H_{[1]22}^{SO} B_3^- \tag{3.10d}$$

as generalizations of (1.4) in the spin–orbit coupling context if

$$H_{SS}^{SO} = H_{\alpha\beta}^{SO} e_{\alpha,\beta} \otimes e_{1,1} + H_{[1]\alpha\beta}^{SO} e_{\alpha,\beta} \otimes e_{2,2}, \qquad \alpha, \beta = 1, 2 \tag{3.11}$$

However, despite of the fact that the spin–orbit coupling procedure can be associated with extended Darboux transformations [see equation (3.10)] in this three-dimensional context, it is not possible here to insert this spin–orbit coupling procedure inside the standard one, in contrast to the two-dimensional results (2.15) and (2.20). Indeed, let us suppose that B_3^- vanishes. From (3.8) and (2.14), it is clear that this Hamiltonian (3.8) is in fact the direct sum of (2.14): ($H^{SO} \equiv H_{11}^{SO}, H_{[1]}^{SO} \equiv H_{[1]22}^{SO}$) and a similar operator to (2.14), but where ω has been replaced by $-\omega$ ($H_{-\omega}^{SO} \equiv H_{22}^{SO}, H_{[1],-\omega}^{SO} \equiv H_{[1]11}^{SO}$): this is the impact (Moshinsky *et al.*, 1990) of the "*antiparticle*" and it is in complete agreement with the fact that the spin–orbit coupling procedure is related to (relativistic) Dirac Hamiltonians (Beckers and Debergh, 1990) [the operator (3.9) coincides with the Dirac Hamiltonian if the mass term is omitted]. So, by exploiting the connection (2.15), we can try to insert the Hamiltonian (3.8) in the Hamiltonian (3.3), where the operators B_l^- will be replaced by D_l^- = (2.16) or whatever. Thus, we propose the identification [analogous to (2.15)]

$$H = H_{11}^{SO}, \qquad H_{[1]33} = H_{22}^{SO}, \qquad H_{[2]33} = H_{[1]22}^{SO}, \qquad H_{[3]} = H_{[1]11}^{SO} \tag{3.12}$$

Moreover, the corresponding Q is now

$$Q = D_l^-(e_{2,1} + e_{3,1} - e_{7,2} + e_{7,3}) + D_l^{-\prime}(e_{5,4} - e_{6,4} - e_{8,5} - e_{8,6}) \tag{3.13}$$

where [compare with (2.16)]

$$D_l^{-\prime} = \frac{1}{\sqrt{2}}(B_1^- - iB_2^-) \tag{3.14}$$

The operator (3.13) cannot be identified with (3.4) ($D_l^{-\prime}$ evidently being different from D_l^- when an interaction is introduced) and consequently we cannot insert the spin–orbit coupling procedure inside the standard one due to the presence of the operators (3.14) (it is *a fortiori* true when B_3^- is not vanishing). In other words, the concept of "*antiparticles*" is present in the spin–orbit coupling procedure, but cannot be recovered in the standard procedure.

ACKNOWLEDGMENT

We cordially thank Prof. J. Beckers for a careful reading of the manuscript.

REFERENCES

Andrianov, A. A., Borisov, N. V., Eides, M. I., and Ioffe, M. V. (1984). *Physics Letters A*, **105**, 19.
Andrianov, A. A., Borisov, N. V., and Ioffe, M. V. (1985). *Physics Letters A*, **109**, 143.
Balantekin, A. B. (1985). *Annals of Physics*, **164**, 277.
Beckers, J., and Debergh, N. (1990). *Physical Review D*, **42**, 1255.
Beckers, J., Dehin, D., and Hussin, V. (1987). *Journal of Physics A*, **20**, 1137.
Darboux, G. (1882). *Comptes Rendus de l'Académie des Sciences*, **94**, 1456.
Matveev, V. B., and Salle, M. A. (1991). *Darboux Transformations and Solitons*, Springer-Verlag, Berlin, and references therein.
Moshinsky, M., Loyola, G., and Szczepaniak, A. (1990). The two-body Dirac oscillator, in *J. J. Giambiagi Festschrift*, H. Falomir *et al.*, eds., World Scientific, Singapore, pp. 324–349.
Schrödinger, E. (1940). *Proceedings of the Royal Irish Academy*, **46A**, 9, 183.
Witten, E. (1981). *Nuclear Physics B*, **188**, 513.